5. Integracija kvadratnog trinoma

U ovoj lekciji rješavamo integrale sljedećih oblika

$$\int \frac{Ax+B}{ax^2+bx+c}\,dx; \quad \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx; \quad \int \sqrt{ax^2+bx+c}\,dx.$$

Kvadratni trinom ax^2+bx+c, koji se pojavljuje u podintegralnoj f-ji uvijek svodimo na "pogodniji" oblik:

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left(x^2+2\cdot x\cdot\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right) =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\underbrace{\frac{c}{a}-\frac{b^2}{4a^2}}_{\in\mathbb{R}}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 \pm d^2\right].$$

Odrediti integrale

a) $\int \frac{dx}{x^2+4x+8}$; b) $\int \frac{7-8x}{2x^3-3x+1}\, dx$;

c) $\int \frac{3x-2}{x^2+6x+9}\, dx$; d) $\int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\, dx$.

Rj. a) $\int \frac{dx}{x^2+4x+8}$ Pro primjetimo da je

$x^2+4x+8 = x^2+2\cdot x\cdot 2+4+4 = (x+2)^2+4$

i da je $d(x+2)=dx$. Prema tome

$$\int \frac{dx}{x^2+4x+8} = \int \frac{d(x+2)}{(x+2)^2+4} = \frac{1}{2} \operatorname{arctg} \frac{x+2}{2} + C$$

Prema formuli 8. $\int \frac{du}{u^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{u}{a} + C$ pri čemu $u=x+2$, $a=2$

b)
$$2x^3-3x+1 = 2\left(x^3-\frac{3}{2}x+\frac{1}{2}\right) = 2\left(x^2-2\cdot x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{1}{2}\right)$$
$$= 2\left[\left(x-\frac{3}{4}\right)^2+\frac{1}{2}-\frac{9}{16}\right] = 2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]$$

$$\int \frac{7-8x}{2x^3-3x+1}\, dx = \int \frac{7-8x}{2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]}\, dx =$$

uvodimo smjenu
$x-\frac{3}{4}=t$
$dx=dt$
$x=t+\frac{3}{4}$
$7-8x=7-8t-6$
$=1-8t$

$$= \frac{1}{2}\int \frac{1-8t}{t^2-\frac{1}{16}}\, dt = \frac{1}{2}\int \frac{dt}{t^2-\frac{1}{16}} - 4\int \frac{t\, dt}{t^2-\frac{1}{16}} =$$

$$= \frac{1}{2}\cdot \frac{1}{2\cdot\frac{1}{4}} \ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 4\int \frac{\frac{1}{2}\, d\left(t^2-\frac{1}{16}\right)}{t^2-1/16} =$$

$$= \ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 2\ln\left|t^2-\frac{1}{16}\right| + C =$$

$$= \ln\left|\frac{x-\frac{3}{4}-\frac{1}{4}}{t-\frac{3}{4}+\frac{1}{4}}\right| - 2\ln\left|(x-\frac{3}{4})^2-\frac{1}{16}\right| + C$$

$$= \ln\left|\frac{x-1}{x-\frac{1}{2}}\right| - 2\ln\left|x^2-\frac{3}{2}x+\frac{1}{2}\right| + C$$

c) $\int \frac{3x-2}{x^2+6x+9}\,dx$ $\qquad x^2+6x+9 = x^2+2\cdot x\cdot 3+3^2 = (x+3)^2$

$$\int \frac{3x-2}{x^2+6x+9}\,dx = \int \frac{3x-2}{(x+3)^2}\,dx = \left| \begin{array}{l} \text{uvodimo smjenu} \\ x+3=t \\ dx=dt \\ x=t-3 \\ 3x-2=3t-9-2=3t-11 \end{array} \right| =$$

$$= \int \frac{3t-11}{t^2}\,dt = \int\left(\frac{3}{t}-\frac{11}{t^2}\right)dt = 3\int\frac{dt}{t} - 11\int t^{-2}\,dt =$$

$$= 3\ln|t| + 11t^{-1} + C = 3\ln|x+3| + \frac{11}{x+3} + C.$$

d) $J = \int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx$

Prvo podjelimo $6x^3-7x^2+3x-1$ sa $2x-3x^2$:

$$(6x^3-7x^2+3x-1) : (-3x^2+2x) = -2x+1$$

$$\begin{array}{l} -\ \ 6x^3 - 4x^2 \\ \hline = \quad -3x^2+3x-1 \\ -\ \ -3x^2+2x \\ \hline = \qquad x-1 \end{array}$$

Prema tome

$$\frac{6x^3-7x^2+3x-1}{2x-3x^2} = -2x+1+\frac{x-1}{2x-3x^2}$$

$$\int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx = \int\left(-2x+1+\frac{x-1}{2x-3x^2}\right)dx =$$

$$= -2\int x\,dx + \int dx + \int \frac{(x-1)\,dx}{2x-3x^2} = -x^2+x+J_1$$

$$J_1 = -\frac{1}{3}\int \frac{(x-1)\,dx}{x^2-\frac{2}{3}x} = -\frac{1}{3}\int \frac{x}{x(x-\frac{2}{3})}\,dx + \frac{1}{3}\int \frac{dx}{x^2-\frac{2}{3}x}$$

$$= \left| \begin{array}{l} x^2-\frac{2}{3}x = x^2-2\cdot x\cdot\frac{2}{6}+\left(\frac{2}{6}\right)^2-\left(\frac{2}{6}\right)^2 \\ \quad = \left(x-\frac{1}{3}\right)^2-\frac{1}{9} \end{array} \right| =$$

$$= -\frac{1}{3}\int \frac{d(x-\frac{2}{3})}{x-\frac{2}{3}} + \frac{1}{3}\int \frac{d(x-\frac{1}{3})}{\left(x-\frac{1}{3}\right)^2-\frac{1}{9}} =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \frac{1}{3}\cdot\frac{1}{2\cdot\frac{1}{3}}\ln\left|\frac{x-\frac{1}{3}-\frac{1}{3}}{x-\frac{1}{3}+\frac{1}{3}}\right| =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \underbrace{\frac{1}{2}\ln\left|\frac{x-\frac{2}{3}}{x}\right|}_{=\frac{1}{2}\ln|x-\frac{2}{3}|-\frac{1}{2}\ln|x|}$$

$$J = C - x^2 + x + \frac{1}{6}\ln\left|x-\frac{2}{3}\right| - \frac{1}{2}\ln|x|$$

traženo rješenje

(#) Odrediti integrale

a) $\displaystyle\int \frac{dx}{\sqrt{x^3-4x-3}}$ b) $\displaystyle\int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}}$

Rj.

a) $x^2-4x-3 = x^2-2\cdot x\cdot 2+2^2-2^2-3=(x-2)^2-7$

$$\int \frac{dx}{\sqrt{x^3-4x-3}} = \int \frac{dx}{\sqrt{(x-2)^2-7}} = \int \frac{d(x-2)}{\sqrt{(x-2)^2-7}} =$$

$$= \ln\left|x-2+\sqrt{(x-2)^2-7}\right| + C$$

Prema formuli 11. $\displaystyle\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+C$ pri čemu je $u=x-2$, $a=-7$

b)

$$9+6x-3x^2 = (-3)\left[x^2-2x-3\right] = (-3)\left(x^2-2\cdot x\cdot 1+1^2-1^2-3\right) =$$

$$=(-3)\left[(x-1)^2-4\right] = 3\left[4-(x-1)^2\right]$$

$$I=\int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}} = \int \frac{(3x-5)\,dx}{\sqrt{3\left[4-(x-1)^2\right]}} = \left| \begin{array}{ll} \text{uvodimo smjenu} & \\ z=x-1 & \\ dz=dx & 3x-5= \\ x=z+1 & =3z+3-5 \\ & =3z-2 \end{array} \right| =$$

$$=\frac{1}{\sqrt{3}}\int \frac{3z-2}{\sqrt{4-z^2}}\,dz = \underbrace{\frac{3}{\sqrt{3}}}_{=\sqrt{3}} \underbrace{\int \frac{z\,dz}{\sqrt{4-z^2}}}_{=I_1} - \frac{2}{\sqrt{3}} \underbrace{\int \frac{dz}{\sqrt{4-z^2}}}_{=I_2}$$

$$I_1 = \int \frac{z\,dz}{\sqrt{4-z^2}} = \left| \begin{array}{l} d(4-z^2) = -2z\,dz \\ z\,dz = -\frac{1}{2}\,d(4-z^2) \end{array} \right| = -\frac{1}{2}\int (4-z^2)^{-\frac{1}{2}}\,d(4-z^2) =$$

$$= -(4-z^2)^{\frac{1}{2}} + C_1$$

$$I_2 = \int \frac{dz}{\sqrt{4-z^2}} = \arcsin \frac{z}{2} + C_2$$

$$I = \sqrt{3}\left(-(4-z^2)^{\frac{1}{2}}\right) - \frac{2}{\sqrt{3}} \arcsin \frac{z}{2} + C =$$

$$= C - \sqrt{3(4-z^2)} - \frac{2}{\sqrt{3}} \arcsin \frac{z}{2} = \left| \; z = x-1 \; \right|$$

$$= C - \sqrt{9+6x-3x^2} - \frac{2}{\sqrt{3}} \arcsin \frac{x-1}{2}$$

(#) Pomoću formule za parcijalnu integraciju $\int u\,dv = uv - \int v\,du$ odrediti integrale

a) $\int \sqrt{t^2+b}\,dt$; b) $\int \sqrt{a^2-t^2}\,dt$

pa uz pomoć dobijenog rezultata izračunati integrale

(i) $\int \sqrt{x^2-3}\,dx$

(ii) $\int \sqrt{x^2+2x+6}\,dx$

(iii) $\int \sqrt{3+4x-x^2}\,dx$

Rj.

a) $$I = \int \sqrt{t^2+b}\,dt = \left| \begin{array}{ll} u = \sqrt{t^2+b} & dv = dt \\ du = \dfrac{t\,dt}{\sqrt{t^2+b}} & v = t \end{array} \right| =$$

$$= t\sqrt{t^2+b} - \int \frac{t^2+b-b}{\sqrt{t^2+b}}\,dt = t\sqrt{t^2+b} - \int \frac{t^2+b}{\sqrt{t^2+b}}\,dt + b\int \frac{dt}{\sqrt{t^2+b}}$$

$$= t\sqrt{t^2+b} - \underbrace{\int \sqrt{t^2+b}\,dt}_{=I} + b\int \frac{dt}{\sqrt{t^2+b}} \quad \Rightarrow$$

$$\Rightarrow \quad 2I = t\sqrt{t^2+b} + b\int \frac{dt}{\sqrt{t^2+b}}$$

$$I = \frac{1}{2}t\sqrt{t^2+b} + \frac{1}{2}b\ln\left|t+\sqrt{t^2+b}\right| + C$$ traženo rješenje ... (*)

b) $$J = \int \sqrt{a^2-t^2}\,dt = \left| \begin{array}{ll} u = \sqrt{a^2-t^2} & dv = dt \\ du = \dfrac{-t}{\sqrt{a^2-t^2}}\,dt & v = t \end{array} \right| =$$

$$= t\sqrt{a^2-t^2} - \int \frac{-t^2 \overset{+a^2-a^2}{}}{\sqrt{a^2-t^2}}\,dt =$$

$$= t\sqrt{a^2-t^2} - \int \frac{a^2-t^2}{\sqrt{a^2-t^2}} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$= t\sqrt{a^2-t^2} - \underbrace{\int \sqrt{a^2-t^2}}_{=J} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$2J = t\sqrt{a^2-t^2} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$J = \int \sqrt{a^2-t^2}\,dt = \frac{1}{2}t\sqrt{a^2-t^2} + \frac{1}{2}a^2 \arcsin\frac{t}{a} + C \quad \ldots(**)$$

Izračunajmo sad date integrale

(i) $\displaystyle\int \sqrt{x^2-3}\,dx = \left| \begin{matrix} \text{po formuli } (*) \\ \text{gdje je} \\ t=x,\ b=-3 \end{matrix} \right| = \frac{x}{2}\sqrt{x^2-3} - \frac{3}{2}\ln\left|x+\sqrt{a^2-3}\right| + C$

(ii) $x^2+2x+6 = x^2+2\cdot x\cdot 1+1^2-1^2+6 = (x+1)^2+5$

$$\int \sqrt{x^2+2x+6}\,dx = \int \sqrt{(x+1)^2+5}\,d(x+1) = \left| \begin{matrix} \text{po formuli } (*) \\ \text{gdje je} \\ t=x+1,\quad b=5 \end{matrix} \right| =$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2+5} + \frac{5}{2}\ln\left|x+1+\sqrt{(x+1)^2+5}\right| + C$$

(iii) $3+4x-x^2 = (-1)(x^2-4x-3) = (-1)(x^2-2\cdot x\cdot 2+2^2-2^2-3) = -\left[(x-2)^2-7\right] = 7-(x-2)^2$

$$\int \sqrt{3+4x-x^2}\,dx = \int \sqrt{7-(x-2)^2}\,d(x-2) = \left| \begin{matrix} \text{po formuli } (**) \\ \text{gdje je } t=x-2,\ a^2=7 \end{matrix} \right| =$$

$$= \frac{x-2}{2}\sqrt{7-(x-2)^2} + \frac{7}{2}\arcsin\frac{x-2}{\sqrt{7}} + C.$$

Zadaci za vježbu

Odrediti integrale

1. $\displaystyle\int \frac{dx}{x^2-x-6}$

2. $\displaystyle\int \frac{dx}{x^2+4x+29}$

3. $\displaystyle\int \frac{dx}{4x-1-4x^2}$

4. $\displaystyle\int \frac{(4x-3)\,dx}{x^2+3x+4}$

5. $\displaystyle\int \frac{(3x+4)\,dx}{x^2+5x}$

6. $\displaystyle\int \frac{18x^2+12x}{1+6x+9x^2}\,dx$

7. $\displaystyle\int \frac{x^3-2x^2+4}{x^2+2x-3}\,dx$

8. $\displaystyle\int \frac{dx}{\sqrt{2+x-x^2}}$

9. $\displaystyle\int \frac{dx}{\sqrt{x^2-2x}}$

10. $\displaystyle\int \frac{(x+3)\,dx}{\sqrt{1-4x^2}}$

11. $\displaystyle\int \frac{(x-3)\,dx}{\sqrt{x^2+6x}}$

12. * $\displaystyle\int \frac{x\,dx}{\sqrt{1-2x-3x^2}}$

13. $\displaystyle\int \sqrt{x^2+4x}\,dx$

14. $\displaystyle\int \sqrt{1-2x-x^2}\,dx$

Rješenja:

1. $\frac{1}{5}\operatorname{arctg}\frac{x+2}{5}$ 2. ? 3. $\frac{1}{4x-2}$ 4. $2\ln(x^2+3x+4)-\frac{18}{\sqrt{7}}\operatorname{arctg}\frac{2x+3}{\sqrt{7}}$

5. $\frac{4}{5}\ln|x|+\frac{11}{5}\ln|x+5|$ 6. $2x+\frac{1}{9}\left(\frac{7}{3x+1}+\ln|3x+1|\right)$ 7. $\frac{1}{2}x^2-4x+$
$+\frac{3}{4}\ln|x-1|+\frac{41}{4}\ln|x+3|$ 8. $\arcsin\frac{2x-1}{3}$ 9. $\ln\left|x-1+\sqrt{x^2-2x}\right|$

10. $\frac{3}{2}\arcsin 2x-\frac{1}{4}\sqrt{1-4x^2}$ 11. $\sqrt{x^2+6x}-6\ln\left|x+3+\sqrt{x^2+6x}\right|$

12. $-\frac{1}{3\sqrt{3}}\arcsin\frac{3x+1}{2}-\frac{1}{3}\sqrt{1-2x-3x^2}$ 13. $\frac{x+2}{2}\sqrt{x^2+4x}-$
$-2\ln\left|x+2+\sqrt{x^2+4x}\right|$ 14. $\frac{x+1}{2}\sqrt{1-2x-x^2}+\arcsin\frac{x+1}{\sqrt{2}}$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija kvadratnog trinoma)

$$\int \frac{dx}{ax^2+b}, \quad \int \frac{dx}{\sqrt{ax^2+b}}, \quad \text{smjena } \sqrt{|a|}\cdot x = \sqrt{|b|}\cdot t$$

1.
$$\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \left| \begin{array}{l} 2x = 3t \\ 2dx = 3dt \\ dx = \frac{3}{2}dt \end{array} \right| = \frac{3}{2}\int \frac{dt}{(3t)^2+3^2} = \frac{3}{2}\int \frac{dt}{9t^2+9} =$$
$$t = \frac{2x}{3}$$
$$= \frac{3}{2}\cdot\frac{1}{9}\int \frac{dt}{t^2+1} = \frac{1}{6}\operatorname{arctg} t + C = \frac{1}{6}\operatorname{arctg}\frac{2x}{3} + C$$

2.
$$\int \frac{dx}{\sqrt{2x^2+25}} = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}} = \left| \begin{array}{l} \sqrt{2}x = 5t \\ \sqrt{2}\,dx = 5dt \\ dx = \frac{5}{\sqrt{2}}\,dt \\ t = \frac{\sqrt{2}}{5}x \end{array} \right| = \frac{5}{\sqrt{2}}\int \frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}} =$$
$$= \frac{5}{\sqrt{2}}\cdot\frac{1}{5}\int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}}\cdot \ln\left|t+\sqrt{t^2+1}\right| + C = \frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{5}x+\sqrt{\frac{2}{25}x^2+1}\right| + C$$

3.
$$\int \frac{dx}{5x^2-49} = \int \frac{dx}{(\sqrt{5}x)^2-7^2} = \left| \begin{array}{l} \sqrt{5}x = 7t \\ \sqrt{5}\,dx = 7dt \\ dx = \frac{7}{\sqrt{5}}\,dt \\ t = \frac{\sqrt{5}x}{7} \end{array} \right| = \frac{7}{\sqrt{5}}\int \frac{dt}{49t^2-49} = \frac{7}{\sqrt{5}}\cdot\frac{1}{49}\int \frac{dt}{t^2-1}$$
$$= \frac{1}{7\sqrt{5}}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{14\sqrt{5}}\ln\left|\frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1}\right| + C = \frac{1}{14\sqrt{5}}\ln\left|\frac{\sqrt{5}x-7}{\sqrt{5}x+7}\right|$$

4.
$$\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}} = \left| \begin{array}{l} 3x = \sqrt{7}t \\ 3dx = \sqrt{7}\,dt \\ dx = \frac{\sqrt{7}}{3}\,dt \\ t = \frac{3x}{\sqrt{7}} \end{array} \right| = \frac{\sqrt{7}}{3}\int \frac{dt}{\sqrt{7-7t^2}} = \frac{\sqrt{7}}{3}\cdot\frac{1}{\sqrt{7}}\int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{3}\arcsin t + C = \frac{1}{3}\arcsin\left(\frac{3x}{\sqrt{7}}\right) + C$$

5. $\int \frac{dx}{4x^2+11}$, Rj. $\frac{\sqrt{11}}{22}\operatorname{arctg}\frac{2\sqrt{11}x}{11} + C$

6. $\int \frac{dx}{\sqrt{9x^2-16}}$, Rj. $\frac{1}{3}\ln\left|3x+\sqrt{9x^2-16}\right| + C$

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad ax^2+bx+c = a(x-\alpha)^2+\beta$$

7. $I = \int \frac{dx}{x^2+6x+13}$, $\quad x^2+6x+13 = x^2+2\cdot x\cdot 3+3^2+4 = (x+3)^2+4$

$$I = \int \frac{dx}{(x+3)^2+2^2} = \left| \begin{array}{l} x+3 = 2t \\ dx = 2dt \\ t = \frac{x+3}{2} \end{array} \right| = 2\int \frac{dt}{4t^2+4} = 2\cdot\frac{1}{4}\int \frac{dt}{t^2+1} = \frac{1}{2}\operatorname{arctg} t + C$$

$$= \frac{1}{2}\operatorname{arctg}\frac{x+3}{2} + C$$

8. $I = \int \frac{dx}{\sqrt{2-x-x^2}}$, $\quad 2-x-x^2 = -x^2-x+2 = (-1)\left[x^2+x-2\right] =$

$$= (-1)\left(x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right) =$$

$$= (-1)\left[\left(x+\frac{1}{2}\right)^2-\frac{9}{4}\right] = \frac{9}{4}-\left(x+\frac{1}{2}\right)^2$$

$$I = \int \frac{dx}{\sqrt{\frac{9}{4}-\left(x+\frac{1}{2}\right)^2}} = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}} = \left| \begin{array}{l} x+\frac{1}{2} = \frac{3}{2}t \\ dx = \frac{3}{2}dt \\ t = \frac{2x+1}{3} \end{array} \right| = \frac{3}{2}\int \frac{dt}{\sqrt{\frac{9}{4}-\frac{9}{4}t^2}} =$$

$$= \frac{3}{2}\cdot\frac{1}{\frac{3}{2}}\int \frac{dt}{\sqrt{1-t^2}} = \frac{3}{2}\cdot\frac{2}{3}\cdot\arcsin t + C = \arcsin\frac{2x+1}{3} + C$$

9. $I = \int \frac{dx}{2x^2-7x+3}$, $\quad 2x^2-7x+3 = 2\cdot\left(x^2-\frac{7}{2}x+\frac{3}{2}\right) = 2\cdot\left(x^2-2\cdot x\cdot\frac{7}{4}+\left(\frac{7}{4}\right)^2-\right.$

$$\left.-\left(\frac{7}{4}\right)^2+\frac{3}{2}\right) = 2\cdot\left[\left(x-\frac{7}{2}\right)^2+\frac{-49+24}{16}\right] = 2\left[\left(x-\frac{7}{2}\right)^2-\frac{25}{16}\right]$$

$$I = \frac{1}{2}\int \frac{dx}{\left(x-\frac{7}{2}\right)^2-\frac{25}{16}} = \frac{1}{2}\int \frac{dx}{\left(x-\frac{7}{2}\right)^2-\left(\frac{5}{4}\right)^2} = \left| \begin{array}{l} x-\frac{7}{4} = \frac{5}{4}t \\ dx = \frac{5}{4}dt \\ t = \frac{4x-14}{5} \end{array} \right| = \frac{1}{2}\cdot\frac{5}{4}\int \frac{dt}{\frac{25}{16}t^2-\frac{25}{16}}$$

$$= \frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{8}}}\cdot\frac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{25}}}\int \frac{dt}{t^2-1} = \frac{2}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}-1}{\frac{4x-2}{5}+1}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}}{\frac{4x-2}{5}}\right| + C = \frac{1}{5}\ln\left|\frac{4x-12}{4x-2}\right| + C = \frac{1}{5}\ln\left|\frac{2x-6}{2x-1}\right| + C$$

10. $\int \frac{dx}{\sqrt{x^2+8x+25}}$, $\quad$ Rj: $\ln\left|\frac{x+4}{3}+\sqrt{\left(\frac{x+4}{3}\right)^2+1}\right| + C$

$$\int \frac{mx+n}{ax^2+bx+c}\,dx, \qquad \int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$$

11.

$$I = \int \frac{x+4}{x^2-4x+5}\,dx$$

$$(x^2-4x+5)' = 2x-4$$

$$x+4 = a\cdot(2x-4)+b, \quad a, b = ?$$

$$x+4 = 2ax - 4a + b$$

$$2a = 1 \qquad -4a+b = 4$$

$$a = \frac{1}{2} \qquad -2+b = 4$$

$$b = 6$$

$$I = \int \frac{\frac{1}{2}(2x-4)+6}{x^2-4x+5}\,dx$$

$$= \frac{1}{2}\underbrace{\int \frac{2x-4}{x^2-4x+5}\,dx}_{I_1} + 6\underbrace{\int \frac{dx}{x^2-4x+5}}_{I_2} = \frac{1}{2}I_1 + 6\,I_2$$

$$x^2-2\cdot x\cdot 2+2^2-2^2+5 = $$
$$= (x-2)^2+1$$

$$I_1 = \left| \begin{matrix} x^2-4x+5 = t \\ (2x-4)dx = dt \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C_1 = \ln|x^2-4x+5| + C_1$$

$$I_2 = \int \frac{dx}{(x-2)^2+1} = \left| \begin{matrix} x-2 = t \\ dx = dt \end{matrix} \right| = \int \frac{dt}{t^2+1} = \operatorname{arc\,tg} t + C_2 =$$
$$= \operatorname{arc\,tg}(x-2) + C_2$$

$$I = \frac{1}{2}\ln|x^2-4x+5| + \operatorname{arc\,tg}(x-2) + C$$

12.

$$I = \int \frac{x}{\sqrt{x^2+2x-5}}\,dx$$

$$(x^2+2x-5)' = 2x+2$$

$$x = a(2x+2)+b$$

$$x = 2ax+2a+b$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

$$2a+b = 0$$

$$2\cdot\frac{1}{2}+b = 0$$

$$b = -1$$

$$I = \int \frac{\frac{1}{2}(2x+2)-1}{\sqrt{x^2+2x-5}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-5}}\,dx - \int \frac{dx}{\sqrt{x^2+2x-5}} = \frac{1}{2}I_1 - I_2$$

$$\int \frac{2x+2}{\sqrt{x^2+2x-5}}\,dx = \left| \begin{matrix} x^2-2x-5 = t \\ (2x-3)dx=dt \end{matrix} \right| = \int \frac{dx}{\sqrt{t}} = \frac{1}{2}\sqrt{t} + C_1 = \frac{1}{2}\sqrt{x^2+2x-5} + C_1$$

$$x^2+2x-5 = x^2+2\cdot x\cdot 1+1^2-1^2-5 =$$
$$= (x+1)^2-6$$

$$I_2 = \int \frac{dx}{\sqrt{(x+1)^2-6}} = \left| \begin{matrix} x+1 = \sqrt{6}\,t \\ dx = \sqrt{6}\,dt \\ t = \frac{x+1}{\sqrt{6}} \end{matrix} \right| =$$

$$= \sqrt{6}\int \frac{dt}{\sqrt{6t^2-6}} = \frac{\sqrt{6}}{\sqrt{6}}\int \frac{dt}{\sqrt{t^2-1}} = \ln\left|t+\sqrt{t^2-1}\right| + C_2$$

$$= \ln\left|\frac{x+1}{\sqrt{6}} + \sqrt{\frac{(x+1)^2}{6}-1}\right| + C_2$$

$$I = \sqrt{x^2+2x-5} - \ln\left|\frac{x+1}{\sqrt{6}} + \sqrt{\frac{(x+1)^2}{6}-1}\right| + C$$

13. $I = \int \frac{6x-7}{x^2-4x+5}\,dx$, $\quad (x^2-4x+5)' = 2x-4$

$6x-7 = a(2x-4)+b$

$6x-7 = 2ax-4a+b$

$2a = 6 \Rightarrow a = 3$

$b-4a = -7 \Rightarrow b = 5$

$x^2-4x+5 = x^2-2\cdot x\cdot 2+2^2+5-2^2 = (x-2)^2+1$

$$I = \int \frac{3(2x-4)+5}{x^2-4x+5}\,dx = 3\int \frac{2x-4}{x^2-4x+5}\,dx + 5\int \frac{dx}{x^2-4x+5} = 3I_1 + 5I_2$$

$$\int \frac{2x-4}{x^2-4x+5}\,dx = \left|\begin{array}{l} x^2-4x+5 = t \\ (2x-4)\,dx = dt \end{array}\right| = \int \frac{dt}{t} = \ln|t| + C_1 = \ln|x^2-4x+5| + C_1$$

$$I_2 = \int \frac{dx}{(x-2)^2+1} = \left|\begin{array}{l} x-2 = t \\ dx = dt \end{array}\right| = \int \frac{dt}{t^2+1} = \operatorname{arc\,tg} t + C_2 =$$

$$= \operatorname{arc\,tg}(x-2) + C_2$$

$$I = 3\ln|x^2-4x+5| + 5\operatorname{arc\,tg}(x-2) + C$$

14. $\int \frac{3x+2}{\sqrt{x^2-8x-9}}\,dx$, $\quad$ Rj: $3\sqrt{x^2-8x-9} + 14\ln\left|\frac{x-4}{5} + \sqrt{\left(\frac{x-4}{5}\right)^2-1}\right| + C$

15. $\int \frac{3x+4}{\sqrt{-x^2+6x-8}}\,dx$, $\quad$ Rj: $-3\sqrt{-x^2+6x-8} + 13\arcsin(x-3)$

16. $\int \frac{2x+3}{\sqrt{4x^2+4x+3}}\,dx$

17. $\int \frac{x-4}{x^2-5x+6}\,dx$

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1} + C, \quad a \neq -1.$

2. $\int u^{-1}\,du = \int \frac{du}{u} = \int \frac{u'}{u}\,dx =$
$= \ln|u| + C.$

3. $\int a^u\,du = \frac{a^u}{\ln a} + C; \int e^u\,du =$
$= e^u + C.$

4. $\int \sin u\,du = -\cos u + C.$

5. $\int \cos u\,du = \sin u + C.$

6. $\int \sec^2 u\,du = \operatorname{tg} u + C.$

7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u + C.$

8. $\int \frac{du}{u^2+a^2} = \frac{1}{a} \operatorname{arc\,tg} \frac{u}{a} + C.$

9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a} \ln \left| \frac{u-a}{u+a} \right| + C.$

10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C.$

11. $\int \frac{du}{\sqrt{u^2+a}} =$
$= \ln \left| u + \sqrt{u^2+a} \right| + C.$